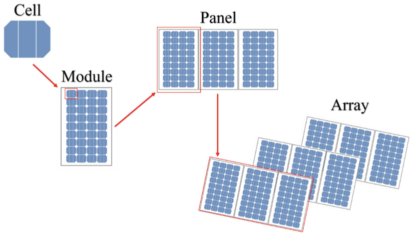
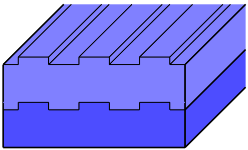
A multiscale hybrid-mixed method for the Helmholtz equation with quasi-periodic boundary conditions

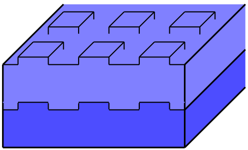
T. Chaumont-Frelet , Z. Kassali and S. Lanteri

Rencontre JCJC Ondes
28th November, 2022

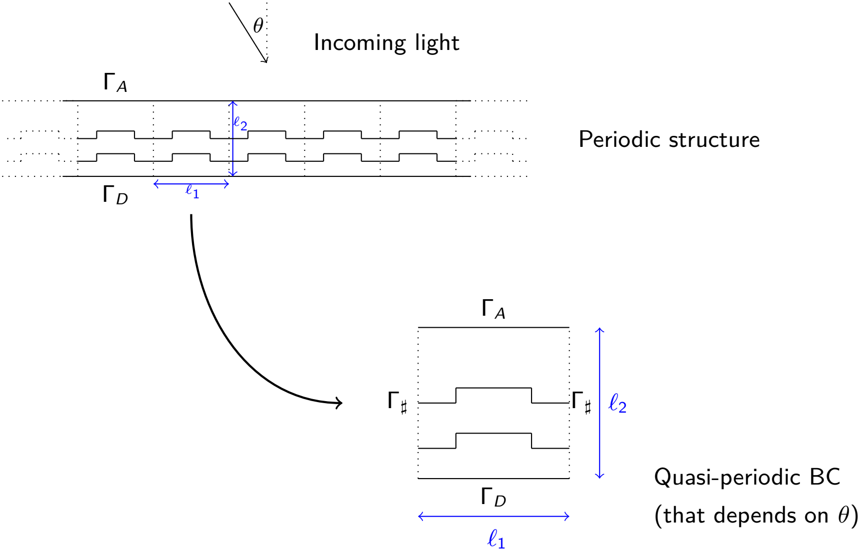
Periodic structure: photovoltaic cells







2D periodic structure



Helmholtz equation

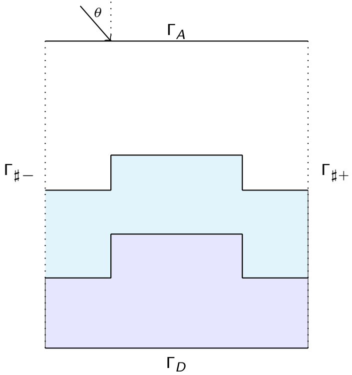
We consider the following Helmholtz problem

$$\begin{cases} -k^2\varepsilon u - \nabla\cdot(\boldsymbol{A}\nabla u) & = & f & \text{in } \Omega, \\ \boldsymbol{A}\nabla u\cdot\boldsymbol{n} - \mathscr{R}u & = & 0 & \text{on } \Gamma_{\mathrm{A}}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1}u_+ & = & 0 & \text{on } \Gamma_\sharp. \end{cases}$$

where

$\Omega := (0,\ell_1)\times(0,\ell_2)$
$k \in \mathbb{R}_+^*$ is the angular frequency
f is a given source term
$\alpha = k\sin(\theta) \in \mathbb{R}$
ε, $\boldsymbol{A}$ are the physical coefficients

Incoming light



Helmholtz equation

Let us assume that $\varepsilon = 1$ and $\boldsymbol{A} = I$

$$
\left\{
\begin{array}{rcll}
-k^2 u - \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u) & = & f & \text{in } \Omega, \\
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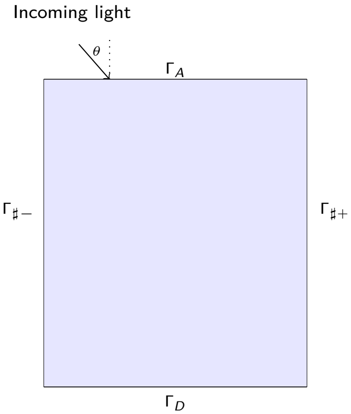
$$



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$\Omega := (0, \ell_1) \times (0, \ell_2)$
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$\alpha = k \sin(\theta) \in \mathbb{R}$

Variational formulation

The standard variational formulation reads: find $u \in H^1_\sharp(\Omega)$ such that

$$b(u, v) = (f, v)_\Omega \quad \forall v \in H^1_\sharp(\Omega),$$

where

$$b(u, v) := -k^2(u, v)_\Omega - \langle \mathscr{R} u, v \rangle_{\Gamma_\mathrm{A}} + (\nabla u, \nabla v)_\Omega,$$

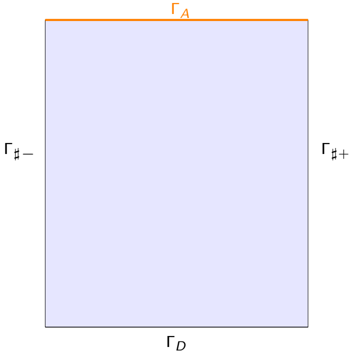
and

$$H^1_\sharp(\Omega) := \left\{ v \in H^1(\Omega) \mid v_{|_{\Gamma_D}} = 0 \ \text{ and } \ v_+ = e^{i\alpha\ell_1} v_- \right\}.$$

- We are investigating the impact of the QPBC.

We consider the following Helmholtz problem

$$\begin{cases} -k^2 u - \nabla \cdot (\nabla u) &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} - \mathscr{R} u &= 0 \quad \text{on } \Gamma_{\mathrm{A}}, \\ u &= 0 \quad \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1} u_+ &= 0 \quad \text{on } \Gamma_{\sharp}. \end{cases}$$



DtN to truncate the unbounded computational

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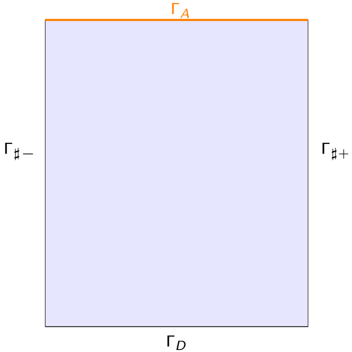

where

$$\mathscr{R} u := \sum_{n \in \mathbb{Z}} i k_n \widehat{u}_n e^{i(\alpha + \alpha_n) x_1},$$

and

$$k_n^2 := k^2 - (\alpha + (2n\pi/\ell_1))^2 .$$

DtN to truncate the unbounded computational

We have

$$\mathscr{R}u := \sum_{n\in\mathbb{Z}} ik_n\widehat{u}_n e^{i(\alpha+\alpha_n)x_1},$$

with

$$k_n^2 := k^2 - (\alpha+\alpha_n)^2.$$

- $\mathscr{R}$ represents a "outgoing" condition.
- $\mathscr{R}$ is a non-local operator.
- $\mathscr{R}$ is not convenient for numerical calculation.
- $\mathscr{R}$ is approximated using a PML.

Study motivations

- We focus on homogeneous media with QPBC.
- This specific periodic geometry can affect:
  1. The used numerical methods;
  2. The Perfectly Matched Layer (PML).
- We seek to have frequency-explicit results.
- We show that the MHM handles efficiently the QPBC.

Outline

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Stability estimate

The stability estimate

$$k^2 \|u\|_{0,\Omega} \leqslant \mathscr{C}_{\mathrm{st}}(k) \|f\|_{0,\Omega}$$

is of paramount importance here, since:

- it is equivalent to the well-posedness of the problem,
- it describes how the solution is controlled by r.h.s. for all frequencies,
- $\mathscr{C}_{\mathrm{st}}(k)$ appears in the error estimate of the used numerical method,
- $\mathscr{C}_{\mathrm{st}}(k)$ appears in the error of the solution obtained by using PML.

Standard stability results

- Considering the standard case (without QPBC)

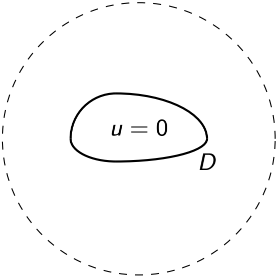
$$\mathscr{C}_{\rm st}(k) \approx k\ell.$$

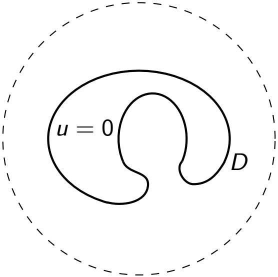
- Some references on stability estimates
  - **F. Ihlenburg, I. Babuska 1997:** one-dimensional problems.
  - **J.M. Melenk 1995:** two-dimensional star-shaped domain.
  - **T. Chaumont-Frelet 2015:** two-dimensional layered media.

Outline

Trapping and non-trapping cases

- $\mathscr{C}_{\mathrm{st}}(k)$ may be linked with the "trapping" characteristics of domain Ω.





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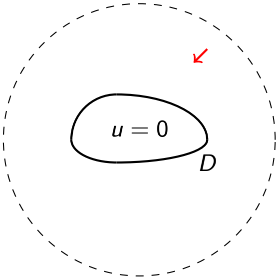

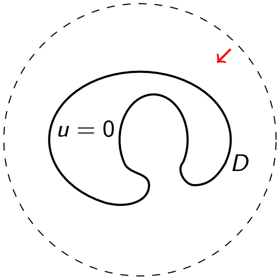

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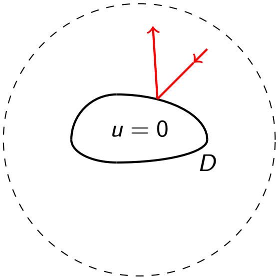

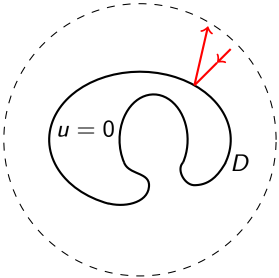

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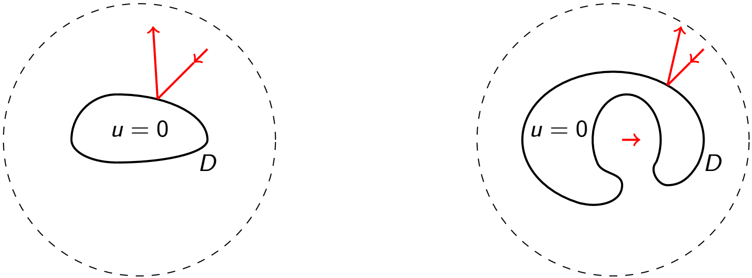

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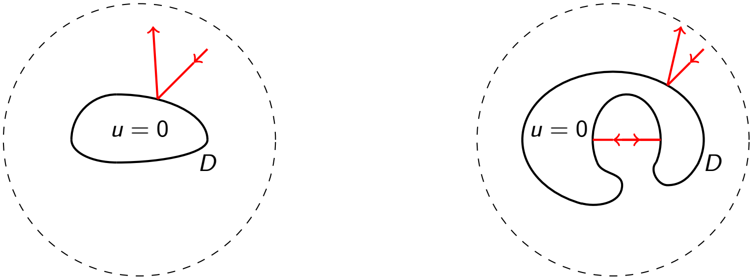

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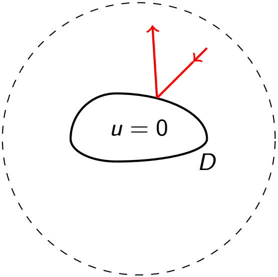

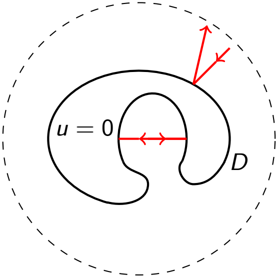

All rays escape Ω.

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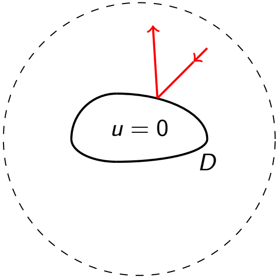

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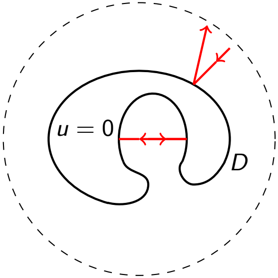

Some rays are trapped.

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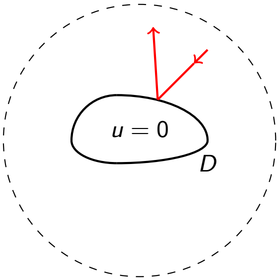

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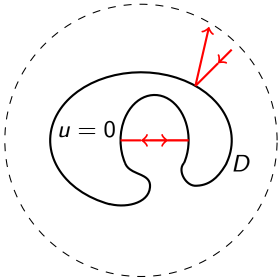

Some rays are trapped.

$$k^2\|u\|_\Omega \lesssim k\ell\|f\|_\Omega$$

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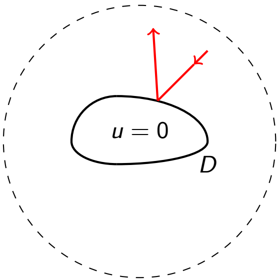

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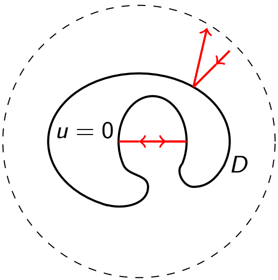
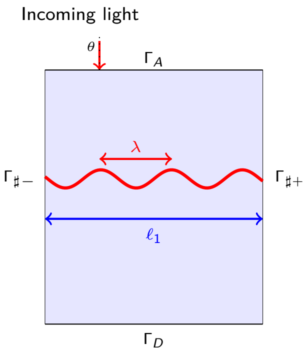


Some rays are trapped.

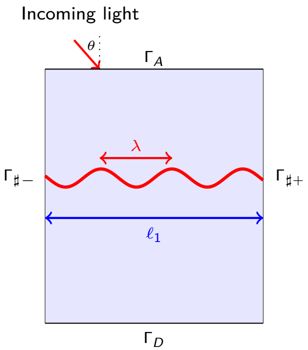
$$k^2\|u\|_\Omega \lesssim \exp(\alpha k\ell)\|f\|_\Omega$$

Quasi-resonant modes



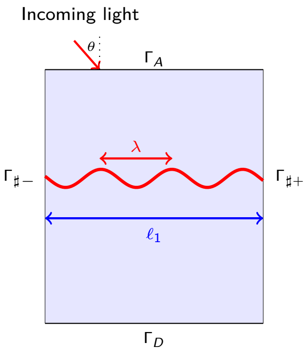
- The quasi-resonance modes are:
  - Plane waves that satisfy the QPBC.
  - Normal incidence: $\exists j \in \mathbb{N}, \qquad \ell_1 k := 2\pi j$.
  -
  -

Quasi-resonant modes



- The quasi-resonance modes are:
  - Plane waves that satisfy the QPBC.
  - Normal incidence: $\exists j \in \mathbb{N}, \qquad \ell_1 k := 2\pi j$.
  - Oblique incidence: $\exists j \in \mathbb{N}, \qquad (1 - \sin(\theta))\, \ell_1 k := 2\pi j$.
  - They depend both on k and θ.

Trapping in periodic case



- For the standard case, without QPBC, we have: $\mathscr{C}_{\mathrm{st}}(k) \approx k\ell$
- What will be the effect of the QPBC? $\mathscr{C}_{\mathrm{st}}(k) \approx ??$

Outline

Stability proof

- Quasi-periodic boundary conditions give that

$$u(\boldsymbol{x}) = \sum_{n\in\mathbb{Z}} \widehat{u}_n(\boldsymbol{x}_2) e^{i(\alpha+\alpha_n)\boldsymbol{x}_1},$$

where

$$\begin{cases} -{k_n}^2 \widehat{u}_n - \widehat{u}_n'' &= \widehat{f}_n \quad \text{on } I = (0, \ell_2), \\ \widehat{u}_n(0) &= 0, \\ \widehat{u}_n'(\ell_2) - i k_n \widehat{u}_n(\ell_2) &= 0, \end{cases}$$

and

$$k_n := \begin{cases} \sqrt{|k^2 - (\alpha+\alpha_n)^2|} & \text{if } (\alpha+\alpha_n)^2 \le k^2 \\ i\sqrt{|k^2 - (\alpha+\alpha_n)^2|} & \text{otherwise.} \end{cases}$$

- We use the notation $k_\star := \min_{n\in\mathbb{Z}} |k_n|$.

Stability proof

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- Using the stability of Fourier modes $\widehat{u}_n$ as

$$\|u\|_{0,\Omega}^2 = \ell_1 \sum_{n\in\mathbb{Z}} \|\widehat{u}_n\|_{0,I}^2.$$

Stability proof: 1D results

Theorem (Estimate for the 1D case)

For all k_n, there exists a unique $\widehat{u}_n$ and we have

$$\|\widehat{u}_n\| \leq 12\min(1, (|k_n|\ell_2)^{-1})\ell_2^2\|\widehat{f}_n\|.$$

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Stability proof: 1D results

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Lemma (One dimensional wave numbers)

We have

$$k_\star\ell_1 = \min_{n\in\mathbb{Z}} |k_n|\ell_1 \geq \Lambda_{k,\boldsymbol{\ell},\theta}\sqrt{k\ell_1}$$

where $\Lambda_{k,\boldsymbol{\ell},\theta} \in [0,1]$ is defined by

$$\Lambda_{k,\boldsymbol{\ell},\theta} := \sqrt{2\pi\min\left(\left\{\frac{k\ell_1(1-\sin\theta)}{2\pi}\right\}, \left\{\frac{k\ell_1(1+\sin\theta)}{2\pi}\right\}\right)},$$

with $\{x\} = \min_{n\in\mathbb{N}} |x-n|$ for $x \in \mathbb{R}$.

Stability results

Theorem (Stability in homogeneous media)

We have

$$\mathscr{C}_{\rm st} \leq 12 \min\left(1, (k_\star \ell_2)^{-1}\right)(k\ell_2)^2.$$

In particular,

$$\mathscr{C}_{\rm st} \leq 12 \min\left(1, \frac{\ell_1}{\ell_2}\frac{1}{\Lambda_{k,\ell,\theta}\sqrt{k\ell_1}}\right)(k\ell_2)^2.$$

- Sharp stability result in homogeneous media :
  - $\mathscr{C}_{\rm st}(k) \approx (k\ell)$ without QPBC.
  - $\mathscr{C}_{\rm st}(k) \approx (k\ell)^2$ if k is close to the resonance frequencies.
  - $\mathscr{C}_{\rm st}(k) \approx (k\ell)^{3/2}$ if k is far from the resonance frequencies.
  - We loose one power of the frequency k.

Outline

Numerical example

Let us consider the following 2D Helmholtz problem

$$\begin{cases} -k^2 u - \Delta u &= f \quad \text{in } (0,1)\times(0,1), \\ \nabla u \cdot \boldsymbol{n} - \mathscr{R} u &= 0 \quad \text{on } \Gamma_{\mathrm{A}}, \\ u &= 0 \quad \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1} u_+ &= 0 \quad \text{on } \Gamma_\sharp. \end{cases}$$

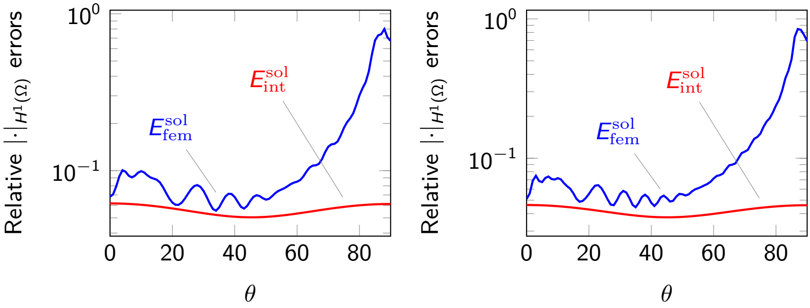
- The source $f \in L^2(\Omega)$ is chosen so that

$$u(\boldsymbol{x}) = \chi(\boldsymbol{x}) e^{ik\boldsymbol{d}_{in}\cdot\boldsymbol{x}} + e^{ik\boldsymbol{d}_{out}\cdot\boldsymbol{x}},$$

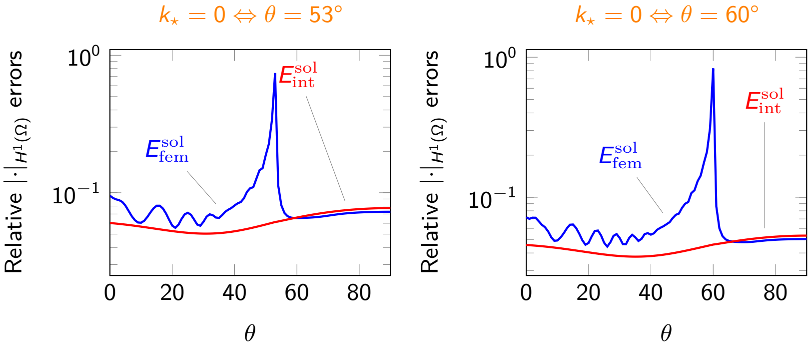
where : $\boldsymbol{d}^{\mathrm{in}} \cdot \boldsymbol{d}^{\mathrm{in}} = \boldsymbol{d}^{\mathrm{out}} \cdot \boldsymbol{d}^{\mathrm{out}} = 1$, $\boldsymbol{d}_1^{\mathrm{in}} = \boldsymbol{d}_1^{\mathrm{out}} = \alpha + m\pi$ for some $m \in \mathbb{N}$, $\boldsymbol{d}_2^{\mathrm{in}} \leq 0$ and $\boldsymbol{d}_2^{\mathrm{out}} = -\boldsymbol{d}_2^{\mathrm{in}}$.

- This solution has only one $k_m = k_\star$.

$k_\star = 0 \Leftrightarrow \theta = 90°$.



$k = 10\pi$ (left) and $k = 15\pi$ (right).



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PML to truncate the unbounded computational domain

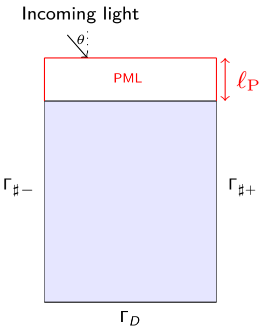
PML problem in $\widetilde{\Omega} := (0, \ell_1) \times (0, \ell_2 + \ell_{\mathrm{P}})$

$$\begin{cases} -k^2 \nu u - \dfrac{\partial}{\partial \boldsymbol{x}_1}\left(\nu \dfrac{\partial u}{\partial \boldsymbol{x}_1}\right) - \dfrac{\partial}{\partial \boldsymbol{x}_2}\left(\nu^{-1} \dfrac{\partial u}{\partial \boldsymbol{x}_2}\right) & = & f & \text{in } \widetilde{\Omega}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{P}}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1} u_+ & = & 0 & \text{on } \widetilde{\Gamma}_{\sharp}. \end{cases}$$

PML parameters

- The depth $\ell_{\mathrm{P}} > 0$.
- The absorption coefficients $\gamma_r, \gamma_i > 0$.
- We further introduce

$$\nu := \begin{cases} 1 & \text{if} & \boldsymbol{x}_2 < \ell_2, \\ \gamma_r + i\gamma_i & \text{in} & \boldsymbol{x}_2 > \ell_2. \end{cases}$$



Outline

PML operator

- We rewrite the problem as

$$\begin{cases} -k^2 u - \Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1} u_+ &= 0 & \text{on } \Gamma_\sharp, \\ \boldsymbol{\nabla} u \cdot \boldsymbol{n} &= \nu^{-1} \boldsymbol{\nabla} u_{\mathrm{P}} \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{A}}, \end{cases}$$

and

$$\begin{cases} -k^2 \nu^2 u_{\mathrm{P}} - \nu^2 \dfrac{\partial^2 u_{\mathrm{P}}}{\partial \boldsymbol{x}_1^2} - \dfrac{\partial^2 u_{\mathrm{P}}}{\partial \boldsymbol{x}_2^2} &= 0 & \text{in } \Omega_{\mathrm{P}}, \\ u_{\mathrm{P}} &= 0 & \text{on } \Gamma_{\mathrm{P}}, \\ u_{\mathrm{P},-} - e^{i\alpha\ell_1} u_{\mathrm{P},+} &= 0 & \text{on } \Gamma_{\mathrm{P},\sharp}, \\ u_{\mathrm{P}} &= u & \text{on } \Gamma_{\mathrm{A}}. \end{cases}$$

- We may define $\mathscr{R}_{\mathrm{P}} : H^{1/2}_\sharp(\Gamma_{\mathrm{A}}) \to \left(H^{1/2}_\sharp(\Gamma_{\mathrm{A}})\right)'$ as

$$\mathscr{R}_{\mathrm{P}} u := \nu^{-1} \boldsymbol{\nabla} u_{\mathrm{P}} \cdot \boldsymbol{n}.$$

PML error results

- We may represent the PML as an explicit operator $\mathscr{R}_{\mathrm{P}}$

$$\mathscr{R}_{\mathrm{P}} U = i \sum_{n\in\mathbb{Z}} \frac{1 + e^{2i\nu k_n \ell_{\mathrm{P}}}}{1 - e^{2i\nu k_n \ell_{\mathrm{P}}}} k_n \widehat{U}_n e^{i(\alpha+\alpha_n)x_1} \qquad \forall U \in H^{1/2}_{\sharp}(\Gamma_{\mathrm{A}}).$$

- We compute the error $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \leq \mathscr{E}_{\mathrm{P}}$, with

$$\mathscr{E}_{\mathrm{P}} := \frac{1}{\gamma_\star \ell_{\mathrm{P}}} \left(1 + \gamma_\star \Lambda_{k,\boldsymbol{\ell},\theta} \sqrt{k\ell_1} \frac{\ell_{\mathrm{P}}}{\ell_1}\right) \exp\left(-\gamma_\star \Lambda_{k,\boldsymbol{\ell},\theta} \sqrt{k\ell_1} \frac{\ell_{\mathrm{P}}}{\ell_1}\right),$$

and $\gamma_\star = \min(\gamma_r, \gamma_i)$.

  - $\mathscr{E}_{\mathrm{P}} \approx \sqrt{k} e^{-2\gamma_\star \sqrt{k} \ell_{\mathrm{P}}}$ if k is far from the resonance frequencies.
  - $\mathscr{E}_{\mathrm{P}} \approx \frac{1}{\gamma_\star \ell_{\mathrm{P}}}$ if k is close to the resonance frequencies.

PML error results

- Frequency-explicit error for the PML solution

$$k^2 \|u - \widetilde{u}\|_{0,\Omega} \leq \frac{\mathscr{C}_{\mathrm{st}}\mathscr{E}_{\mathrm{P}}}{1 - \mathscr{C}_{\mathrm{st}}\mathscr{E}_{\mathrm{P}}} \mathscr{C}_{\mathrm{st}} \|f\|_{0,\Omega},$$

- In the "standard case" we always have exponential convergence on both ℓ_{P} and $\gamma_\star$.
- Close to the quasi-resonance frequencies, one may only have linear convergence.
- Increasing the coefficient γ_i affect the absorption of the outgoing propagative modes (real k_n).
- Increasing the coefficient γ_r affect the absorption of the outgoing evanescent modes (pure imaginary k_n).

Outline

Let us consider the following 2D Helmholtz problem

$$\begin{cases} -k^2 u - \Delta u &= f \quad \text{in } (0,1)\times(0,1), \\ \nabla u \cdot \boldsymbol{n} - \mathscr{R} u &= 0 \quad \text{on } \Gamma_{\mathrm{A}}, \\ u &= 0 \quad \text{on } \Gamma_{\mathrm{D}}, \\ u_- - e^{i\alpha\ell_1} u_+ &= 0 \quad \text{on } \Gamma_{\sharp}. \end{cases}$$

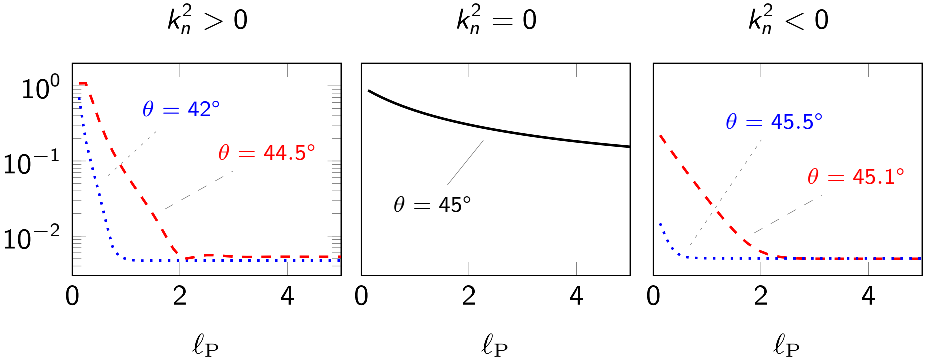
- The source $f \in L^2(\Omega)$ is chosen so that

$$u(\boldsymbol{x}) = \chi(\boldsymbol{x}) e^{ik\boldsymbol{d}_{in}\cdot\boldsymbol{x}} + e^{ik\boldsymbol{d}_{out}\cdot\boldsymbol{x}},$$

  where : $\boldsymbol{d}^{\mathrm{in}} \cdot \boldsymbol{d}^{\mathrm{in}} = \boldsymbol{d}^{\mathrm{out}} \cdot \boldsymbol{d}^{\mathrm{out}} = 1$, $\boldsymbol{d}_1^{\mathrm{in}} = \boldsymbol{d}_1^{\mathrm{out}} = \alpha + n\pi$ for some $n \in \mathbb{N}$, $\boldsymbol{d}_2^{\mathrm{in}} \leq 0$ and $\boldsymbol{d}_2^{\mathrm{out}} = -\boldsymbol{d}_2^{\mathrm{in}}$.

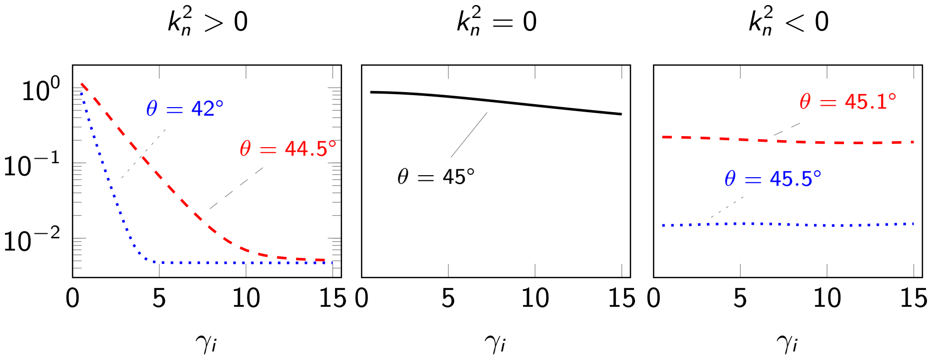
- This solution has only one $k_n = k_\star$.

Error curves for $k = 6.8284\pi$, $H = \frac{1}{256}$, $\gamma_r = 1$ and $\gamma_i = 0.6$.



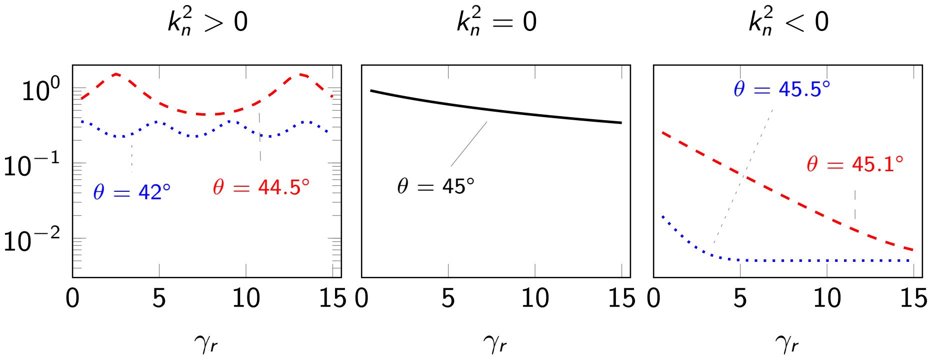
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \sqrt{k_\star} e^{-2\sqrt{k_\star}\ell_{\mathrm{P}}}$ if k is far from the resonance frequency.
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \frac{1}{\ell_{\mathrm{P}}}$ if k is close the resonance frequency.

Error curves for $k = 6.8284\pi$, $H = \frac{1}{256}$, $\ell_{\mathrm{P}} = 1/8$ and $\gamma_r = 1$.



- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \sqrt{k_\star} e^{-2\sqrt{k_\star}\gamma_i \ell_{\mathrm{P}}}$ if k is far from the resonance frequency and $k_n^2 > 0$.
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \frac{1}{\gamma_i}$ if k is close the resonance frequency.
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx C$ if $k_n^2 < 0$.

Error curves for $k = 6.8284\pi$, $H = \frac{1}{256}$, $\ell_{\mathrm{P}} = 1/8$ and $\gamma_i = 1$.



- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx C$ if $k_n^2 > 0$.
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \frac{1}{\gamma_r}$ if k is close the resonance frequency.
- $|(\mathscr{R} - \mathscr{R}_{\mathrm{P}})| \approx \sqrt{k_\star} e^{-2\sqrt{k_\star}\gamma_r \ell_{\mathrm{P}}}$ if k is far from the resonance frequency and $k_n^2 < 0$.

Outline

Stability of the PML problem

- We have shown that the PML problem is well-posed for right-hand sides $f \in L^2(\Omega)$ supported in the original domain.

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- Stability of the problem for general right-hand sides $\widetilde{f} \in L^2(\widetilde{\Omega})$ is required for the stability and convergence of FEM.

- The PML problem: find $\widetilde{u} \in H^1_\sharp(\widetilde{\Omega})$ s.t

$$-k^2(\nu\widetilde{u}, \widetilde{v})_{\widetilde{\Omega}} + (\nu\partial_1\widetilde{u}, \partial_1\widetilde{v})_{\widetilde{\Omega}} + (\nu^{-1}\partial_2\widetilde{u}, \partial_2\widetilde{v})_{\widetilde{\Omega}} = (\widetilde{f}, \widetilde{v})_{\widetilde{\Omega}} \quad \forall \widetilde{v} \in H^1_\sharp(\widetilde{\Omega}),$$

where

$$H^1_\sharp(\widetilde{\Omega}) := \left\{ \widetilde{v} \in H^1(\widetilde{\Omega}) \mid \widetilde{v}_{|_{\Gamma_D}} = \widetilde{v}_{|_{\Gamma_P}} = 0 \text{ and } \widetilde{v}_+ = e^{i\alpha\ell_1}\widetilde{v}_- \right\}.$$

Stability results

The stability estimate

$$k^2 \left\|\widetilde{u}\right\|_{0,\widetilde{\Omega}} \leqslant \widetilde{\mathscr{C}}_{\mathrm{st}}(k) \left\|\widetilde{f}\right\|_{0,\widetilde{\Omega}}$$

Theorem (Stability in homogeneous media)

We have

$$\widetilde{\mathscr{C}}_{\mathrm{st}} \lesssim \min\left(1, (k_\star \ell_2)^{-1}\right)(k\ell_2)^2.$$

In particular,

$$\widetilde{\mathscr{C}}_{\mathrm{st}} \lesssim \min\left(1, \frac{\ell_1}{\ell_2}\frac{1}{\Lambda_{k,\boldsymbol{\ell},\theta}\sqrt{k\ell_1}}\right)(k\ell_2)^2.$$

Outline

Outline

MHM references:

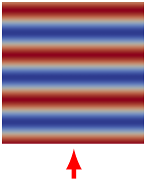
Harder, Christopher and Paredes, Diego and Valentin, Frédéric.
A family of multiscale hybrid-mixed finite element methods for the Darcy equation with rough coefficients.
Journal of Computational Physics, 245: 107-130, 2013.

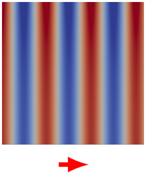
Chaumont-Frelet, Théophile and Valentin, Frédéric.
A multiscale hybrid-mixed method for the Helmholtz equation in heterogeneous domains.
SIAM Journal on Numerical Analysis, 58(2): 1029-1067, 2020.

Motivation

Considering a Cartesian mesh

- First-order polynomial basis functions, the MHM method is exact for plane waves with $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$ direction.
- Quasi-resonant modes are plane waves traveling in one direction.





PML problem in $\widetilde{\Omega} := (0, \ell_1) \times (0, \ell_2 + \ell_{\mathrm{P}})$

We rewrite the PML problem as

$$\left\{ \begin{array}{rcll} -k^2 \nu u - \boldsymbol{\nabla} \cdot (\boldsymbol{D} \boldsymbol{\nabla} u) & = & f & \text{in } \widetilde{\Omega}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{P}}, \\ u_- - e^{i\alpha\ell_1} u_+ & = & 0 & \text{on } \Gamma_\sharp. \end{array} \right.$$

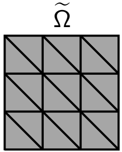
where

$$\boldsymbol{D} = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \quad \text{and} \quad \nu := \left\{ \begin{array}{rcl} 1 & \text{if} & \boldsymbol{x}_2 < \ell_2, \\ \gamma_r + i\gamma_i & \text{in} & \boldsymbol{x}_2 > \ell_2. \end{array} \right.$$

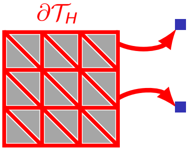
Two challenges

- We employ the MHM method for Helmholtz considering:
  - A quasi-periodic boundary condition.
  - A perfectly matched layer (PML).
- Our goal is to show that the MHM method is robust in the presence of quasi-resonances.

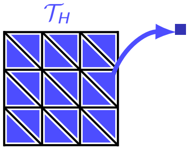
Outline



- The MHM method is based on the primal hybrid formulation of the Helmholtz equation.



- The continuity of the unknown is relaxed on the mesh skeleton.
- Lagrange multipliers derive to weakly ensure the continuity of the solution.



- Independent local problems provide basis functions leading to a multiscale strategy.

Broken space and Lagrange multipliers

Broken space:

$$V = \left\{ v \in L^2(\widetilde{\Omega}) \; ; \; v \in H^1(K) \text{ for all } K \in \mathcal{T}_H \right\}.$$

To weakly enforce the continuity of the solution u we define the space of Lagrange multipliers:

$$\Lambda = \left\{ \mu \in \prod_{K \in \mathcal{T}_H} H^{-1/2}(K) \mid \; \exists \boldsymbol{q} \in \boldsymbol{H}(div; \widetilde{\Omega}) \; \middle| \; \begin{array}{l} \mu|_{\partial K} = \boldsymbol{q} \cdot \boldsymbol{n}_K \;\; \forall K \in \mathcal{T}_H \\ \boldsymbol{q}_+ \cdot \boldsymbol{n}_+ + e^{i\alpha\ell_1} \boldsymbol{q}_- \cdot \boldsymbol{n}_- = 0 \end{array} \right\}.$$

- The effect of the quasi-periodic boundary condition appears in the definition of Λ.

Hybrid formulation

Consider the continuous sesquilinear form $a : V \times V \to \mathbb{C}$

$$a(u, v) = \sum_{K \in \mathcal{T}_H} -k^2 (\nu u, v)_K + (\boldsymbol{D}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_K ,$$

and the pairing $b : \Lambda \times V \to \mathbb{C}$

$$b(\mu, v) = \sum_{K \in \mathcal{T}_H} \langle \mu, v \rangle_{\partial K} .$$

We obtain the key property

$$H^1_\sharp(\widetilde{\Omega}) := \{ v \in V \ | \ b(\mu, v) = 0 \ \ \forall \mu \in \Lambda \} .$$

Hybrid formulation

Hybrid formulation

There exists a unique couple $(\lambda, u) \in \Lambda \times V$ such that:

$$\left\{ \begin{array}{rcll} a(u,v) + b(\lambda, v) & = & (f, v)_{\widetilde{\Omega}} & \forall v \in V, \\ b(\mu, u) & = & 0 & \forall \mu \in \Lambda. \end{array} \right.$$

Rewriting the first equation of the hybrid formulation as

$$a(u,v) \quad = \quad (f, v)_{\widetilde{\Omega}} - \; b(\lambda, v) \qquad \forall v \in V$$

Hybrid formulation

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$$\begin{cases} a(u,v) + b(\lambda, v) &= (f,v)_{\widetilde{\Omega}} \quad \forall v \in V, \\ b(\mu, u) &= 0 \quad \forall \mu \in \Lambda. \end{cases}$$

Rewriting the first equation of the hybrid formulation as

$$\begin{aligned} a(u,v) &= (f,v)_{\widetilde{\Omega}} - b(\lambda, v) \qquad \forall v \in V \\ &= a(\hat{T} f, v) \end{aligned}$$

Hybrid formulation

Hybrid formulation

There exists a unique couple $(\lambda, u) \in \Lambda \times V$ such that:

$$\begin{cases} a(u,v) + b(\lambda, v) = (f, v)_{\widetilde{\Omega}} & \forall v \in V, \\ b(\mu, u) = 0 & \forall \mu \in \Lambda. \end{cases}$$

Rewriting the first equation of the hybrid formulation as

$$\begin{aligned} a(u,v) &= (f,v)_{\widetilde{\Omega}} - b(\lambda, v) & \forall v \in V \\ &= a(\hat{T} f, v) + a(T\lambda, v) & \forall v \in V \end{aligned}$$

If there exist two operators $T : \Lambda \to V$ and $\hat{T} : L^2(\widetilde{\Omega}) \to V$, such that

$$a(\hat{T}f, v) = (f, v)_{\widetilde{\Omega}} \ \ \forall v \in V,$$

and

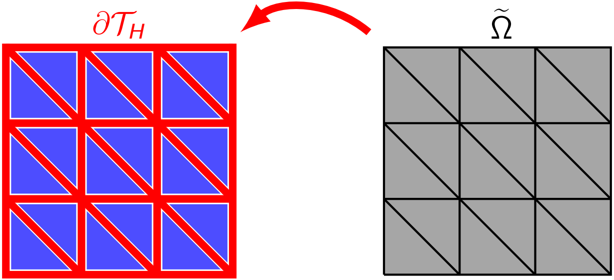
$$a(T\lambda, v) = -b(\lambda, v) \ \ \forall v \in V,$$

then we must have : $u = \hat{T}f + T\lambda.$

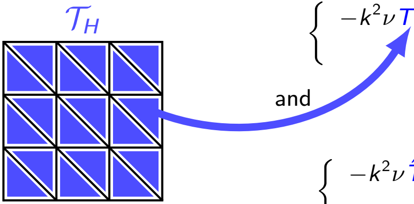
Under the assumption that T and $\hat{T}$ exist, we obtain the MHM formulation:

MHM formulation

$$\begin{cases} \text{Find } \lambda \in \Lambda \text{ such that:} \\ b(\mu, T\lambda) = -b(\mu, \hat{T}f) \qquad \forall \mu \in \Lambda. \end{cases}$$



Using the definition of T and $\hat{T}$ we find that they are solution to the following **local problems** in each $K \in \mathcal{T}_H$:



$$\left\{ \begin{array}{rcll} -k^2\nu T_K\lambda - \boldsymbol{\nabla}\cdot(\boldsymbol{D}\boldsymbol{\nabla} T_K\lambda) & = & 0 & \text{in } K, \\ \boldsymbol{D}\boldsymbol{\nabla}(T_K\lambda)\cdot\boldsymbol{n} & = & -\lambda & \text{on } \partial K, \end{array} \right.$$

and

$$\left\{ \begin{array}{rcll} -k^2\nu \hat{T}_K f - \boldsymbol{\nabla}\cdot\left(\boldsymbol{D}\boldsymbol{\nabla} \hat{T}_K f\right) & = & f & \text{in } K, \\ \boldsymbol{D}\boldsymbol{\nabla}\left(\hat{T}_K f\right)\cdot\boldsymbol{n} & = & 0 & \text{on } \partial K. \end{array} \right.$$

- These local problems can be efficiently solved in parallel.

Local problems

- The operators T and $\hat{T}$ are defined locally in each element thanks to the same local problems as in the usual case (without quasi-periodic conditions).

- The well-posedness of the MHM formulation is equivalent to the well-posedness of the local problems.

- The local problems are well-posed if K is convex and:

$$kH \lesssim |\nu|^{-1}\pi,$$

where $|\nu|$ is the PML dumping parameter.

- Taking into account the above condition, we need slightly finer mesh in the PML.

Outline

MHM stability results

- For the standard case without QPBC, we have $\mathscr{C}_{\mathrm{st}}(k) \approx k\ell$ and

$$\text{MHM and FEM are stable if} \qquad k^2 H \leq C.$$

- Since $\mathscr{C}_{\mathrm{st}}(k) \approx (k\ell)^2$, the "naive" proof shows that:

$$\text{MHM and FEM are stable if} \qquad k^3 H \leq C.$$

- We develop a finer result showing that:

$$\text{MHM is stable if} \qquad k^2 H \leq C.$$

Outline

Elements of the proof: a splitting

Quasi-periodic boundary conditions give that

$$u(\boldsymbol{x}) = \sum_{n\in\mathbb{Z}} \widehat{u}_n(\boldsymbol{x}_2) e^{i(\alpha+\alpha_n)\boldsymbol{x}_1},$$

where $$\left\{ \begin{array}{rcl} -k_n^2\widehat{u}_n - \widehat{u}_n'' &=& \widehat{f}_n \quad \text{on } I = (0,\ell_2), \\ \widehat{u}_n(0) &=& 0, \\ \widehat{u}_n'(\ell_2) - ik_n\widehat{u}_n(\ell_2) &=& 0. \end{array} \right.$$

Using the above expansion, we split the solution as $u = \tilde{u} + \breve{u}$, where

$$\tilde{u} = \sum_{2|k_n|^2 \geq k^2} \widehat{u}_n e^{i(\alpha+\alpha_n)\boldsymbol{x}_1} \quad \text{and} \quad \breve{u} = \sum_{2|k_n|^2 < k^2} \widehat{u}_n e^{i(\alpha+\alpha_n)\boldsymbol{x}_1},$$

which are respectively solutions of the the Helmholtz problem with

$$\tilde{f} = \sum_{2|k_n|^2 \geq k^2} \widehat{f}_n e^{i(\alpha+\alpha_n)\boldsymbol{x}_1} \quad \text{and} \quad \breve{f} = \sum_{2|k_n|^2 < k^2} \widehat{f}_n e^{i(\alpha+\alpha_n)\boldsymbol{x}_1}.$$

For $\tilde{u} = \sum_{2|k_n|^2 \geq k^2} \widehat{u}_n e^{i(\alpha+\alpha_n)x_1}$

- We have $\|\widehat{u}_n\|_{0,I} \leq \frac{1}{|k_n|^2}\|f_n\|_{0,I}$, so that

$$k^2\|\widetilde{u}\|_{0,\widetilde{\Omega}} \lesssim k\ell_2\|\widetilde{f}\|_{0,\widetilde{\Omega}}$$

- **Raviart and Thomas** interpolant convergence result

$$\begin{aligned}\|\tilde{u} - \tilde{u}_H\|_{V,k} &\lesssim H\|\tilde{u}\|_{2,\widetilde{\Omega}} \\ &\lesssim Hk\ell_2\|\tilde{f}\|_{0,\widetilde{\Omega}}.\end{aligned}$$

- It is the usual proof when $\mathscr{C}_{\mathrm{st}}(k) \approx k\ell$.

Elements of proof: the resonant part

For $\breve{u} = \sum_{2|k_n|^2 < k^2} \widehat{u}_n e^{i(\alpha+\alpha_n)\mathbf{x}_1}$

- We have

$$k^2 \|\widetilde{u}\|_{0,\widetilde{\Omega}} \lesssim (k\ell_2)^2 \|\widetilde{f}\|_{0,\widetilde{\Omega}}$$

- **Raviart and Thomas** interpolant convergence result

$$\begin{aligned} \|\tilde{u} - \tilde{u}_H\|_{V,k} &\lesssim H \|\tilde{u}\|_{2,\widetilde{\Omega}} \\ &\lesssim H (k\ell_2)^2 \|\tilde{f}\|_{0,\widetilde{\Omega}}. \end{aligned}$$

Elements of proof: the resonant part

For $\breve{u} = \sum_{2|k_n|^2 < k^2} \widehat{u}_n e^{i(\alpha+\alpha_n)x_1}$

- We employ the following fine stability estimate:

$$\|\widehat{u}_n'\|_{\infty,I} \lesssim k\|f_n\|_{0,I} \quad \text{and} \quad \|\widehat{u}_n'\|_{0,I} \lesssim k\|f_n\|_{0,I},$$

and

$$\begin{aligned} \|\breve{u} - \breve{u}_H\|_{V,k} &\lesssim H \left\| \frac{\partial^2 \breve{u}}{\partial x_1 \partial x_2} \right\|_{0,\widetilde{\Omega}} \\ &\lesssim H k \ell_2 \|\breve{f}\|_{0,\widetilde{\Omega}}. \end{aligned}$$

T. Chaumont-Frelet, D. Paredes, and F. Valentin.
Flux approximation on unfitted meshes and application to MHM methods
(2022, hal-id: 03834748).

Outline

Outline

FEM/MHM stability results

- MHM stability condition

MHM is stable if

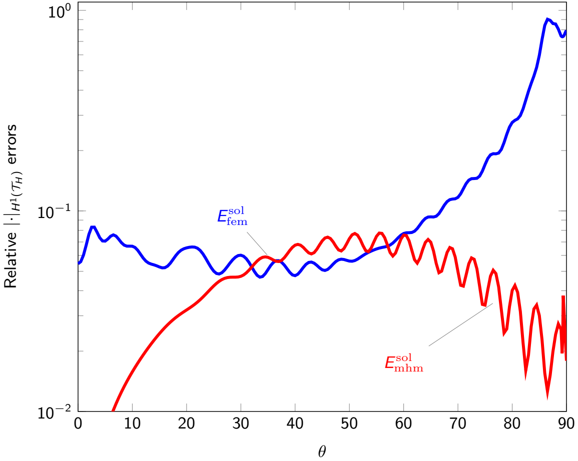
$$k^2 H \leq C.$$

- Classical FEM stability condition

FEM is stable if

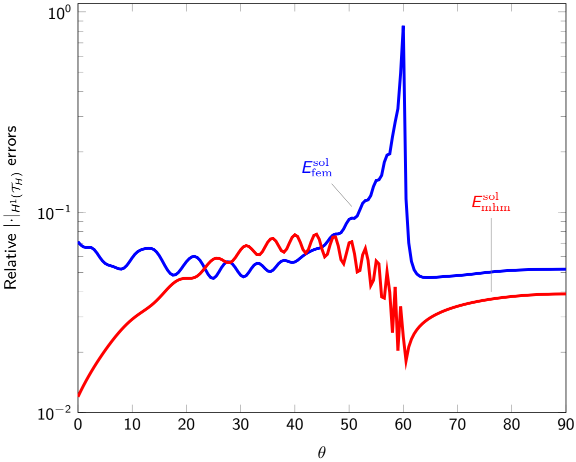
$$k^3 H \leq C.$$

Numerical example 1



$k = 15\pi$ and $H = \frac{1}{256}$, resonance on $\theta = 90$.

Numerical example 2



$k = 15\pi$ and $H = \frac{1}{256}$, resonance on $\theta \approx 60$.

Summary and future works

Summary

- We obtain an optimal frequency-explicit stability estimate.
- We identify the quasi-resonances effect on:
  - The problem stability
  - The numerical method stability
  - The PML technique
- The MHM formulation of Helmholtz equation with
  - The quasi-periodic BC
  - The PML technique
- The MHM is robust with respect to the quasi-resonances.

Thank you!